

HG Nov 11

Answers to three exercises in the lecture notes to Rice chapter 8

Exercise 1 (page 2)

- (i) If $X \sim \text{bin}(n, p)$, we have according to the general theory for the binomial distr. that, for large n , $X \stackrel{\text{approx.}}{\sim} N(E(X), \text{var}(X)) = N(np, np(1-p))$, or

$$\frac{X - np}{\sqrt{np(1-p)}} \stackrel{\text{approx.}}{\sim} N(0, 1)$$

But

$$\frac{X - np}{\sqrt{np(1-p)}} = \frac{n(X/n - p)}{\sqrt{n}\sqrt{p(1-p)}} = \sqrt{n} \frac{\hat{p} - p}{\sqrt{p(1-p)}} \stackrel{\text{approx.}}{\sim} N(0, 1)$$

Hence,

$$b = \sqrt{p(1-p)} \quad \text{and} \quad \hat{b} = \sqrt{\hat{p}(1-\hat{p})}$$

where $\hat{p} = X/n$ is consistent, which implies by continuity that also \hat{b} is consistent for b .

- (ii) According to theory, if $X \sim \text{poisson}(t\lambda)$, then $X \stackrel{\text{approx.}}{\sim} N(E(X), \text{var}(X)) = N(\lambda t, \lambda t)$ for large t (i.e., such that $t\lambda \geq 10$, say). As in (i) we get

$$\frac{X - n\lambda}{\sqrt{n\lambda}} = \frac{n(X/n - \lambda)}{\sqrt{n}\sqrt{\lambda}} = \sqrt{n} \frac{\hat{\lambda} - \lambda}{\sqrt{\lambda}} \stackrel{\text{approx.}}{\sim} N(0, 1)$$

Hence $b = \sqrt{\lambda}$ and $\hat{b} = \sqrt{\hat{\lambda}} = \sqrt{\frac{X}{t}}$ is consistent for b .

- (iii) According to the central limit theorem, for large n ,

$$\bar{X} \stackrel{\text{approx.}}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right) = N(E(\bar{X}), \text{var}(\bar{X})).$$

Hence $b = \sigma$, and, for example, $\hat{b} = S = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}$ is consistent for b .

Exercise 2 (page 11)**(a)**

From $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ and $X' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$, we get

$$X'X = \begin{pmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{pmatrix}$$

from which the determinant follows directly.

(b)

The inverse of a 2×2 - matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is in general given by

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (\text{check by multiplying } A \text{ and } A^{-1})$$

where $D = ad - bc$ is the determinant of A , which must be different from 0 for the inverse to exist.

Hence, from (a):

$$(X'X)^{-1} = \frac{1}{n \sum_i (x_i - \bar{x})^2} \cdot \begin{pmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{pmatrix}$$

Since $\text{cov}(\hat{\beta}) = \sigma^2 (X'X)^{-1}$, we find the variances on the main diagonal, giving the expressions in the exercise.

(c) From the matrices in (a) and (b) we get, writing $D = n \sum_i (x_i - \bar{x})^2$ for the determinant,

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X'X)^{-1} X'Y = \frac{1}{D} \cdot \begin{pmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{pmatrix} \cdot \begin{pmatrix} \sum_i Y_i \\ \sum_i x_i Y_i \end{pmatrix}$$

We now find $\hat{\beta}_1$ as the second element in the vector $\hat{\beta}$:

$$\begin{aligned}\hat{\beta}_1 &= \frac{1}{D} \cdot \left(-\sum_i x_i \sum_i Y_i + n \sum_i x_i Y_i \right) = \frac{1}{D} \cdot \left(n \sum_i x_i Y_i - n^2 \bar{x} \bar{Y} \right) = \\ &= \frac{n \left(\sum_i x_i Y_i - n \bar{x} \bar{Y} \right)}{n \sum_i (x_i - \bar{x})^2} = \frac{\sum_i (x_i - \bar{x})(Y_i - \bar{Y})}{\sum_i (x_i - \bar{x})^2} = \frac{S_{xY}}{S_x^2}\end{aligned}$$

Exercise 3 (page 12)

Since the determinant of the covariance matrix $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ is $D = \sigma_{11}\sigma_{22}(1 - \rho^2)$,

we get the inverse as in exercise 2b

$$\Sigma^{-1} = \frac{1}{D} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

and

$$\begin{aligned}(x - \mu)' \Sigma^{-1} (x - \mu) &= \frac{1}{D} (x_1 - \mu_1, x_2 - \mu_2) \cdot \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \cdot (x - \mu) = \\ &= \frac{1}{D} \left[(x_1 - \mu_1)\sigma_{22} - (x_2 - \mu_2)\sigma_{12}, \quad -(x_1 - \mu_1)\sigma_{12} + (x_2 - \mu_2)\sigma_{11} \right] \cdot \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} = \\ &= \frac{1}{D} \left[(x_1 - \mu_1)^2 \sigma_{22} - (x_2 - \mu_2)(x_1 - \mu_1)\sigma_{12} - (x_1 - \mu_1)(x_2 - \mu_2)\sigma_{12} + (x_2 - \mu_2)^2 \sigma_{11} \right] = \\ &= \frac{1}{\sigma_{11}\sigma_{22}(1 - \rho^2)} \left[(x_1 - \mu_1)^2 \sigma_{22} - 2(x_1 - \mu_1)(x_2 - \mu_2)\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho + (x_2 - \mu_2)^2 \sigma_{11} \right] = \\ &= \frac{1}{1 - \rho^2} \left[\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right) \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right) + \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right)^2 \right]\end{aligned}$$

Multiplying this by $-\frac{1}{2}$, we see that the exponent in (12) reduces exactly to the exponent in Example F in Rice section 3.3. We also see that the expression in the denominator in the pdf in example F (Rice) is the same as in (12) from the expression of the determinant D .